Fundamental Mathematics

Sittisak Leelahanon

Lecture 1
Intensive Course Part I: Mathematics
Ph.D. Program in Business Administration

Number System

1. Natural Number $\mathbb{N}$
$$\mathbb{N} = \{1, 2, 3, \ldots\}$$
2. Integer $\mathbb{Z}$
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \supset \mathbb{N}$$
3. Rational Number $\mathbb{Q}$
$$\mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; m \in \mathbb{Z},\ n \in \mathbb{Z} \setminus \{0\} \right\} \supset \mathbb{Z}$$
4. Real Number $\mathbb{R}$
$$\mathbb{R} = \text{ Completion of } \mathbb{Q} \supset \mathbb{Q}$$

- Irrational Number $\mathbb{Q}^c$ is the real number which is not rational number
$$\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R} \text{ and } \mathbb{Q} \cap \mathbb{Q}^c = \emptyset \text{ and } \mathbb{Q} \cup \mathbb{Q}^c = \mathbb{R}$$

Axioms for Addition

(A) $(\mathbb{R}, +)$ is a commutative group under addition

(A1) Closed under addition

$$\text{For any } x \in \mathbb{R} \text{ and } y \in \mathbb{R}, \quad x + y \in \mathbb{R}$$

(A2) Associative under addition

$$\text{For any } x \in \mathbb{R}, y \in \mathbb{R} \text{ and } z \in \mathbb{R}, \quad (x + y) + z = x + (y + z)$$

(A3) Identity of addition

$$\text{There exists } 0 \in \mathbb{R} \text{ such that } 0 + x = x = x + 0 \text{ for all } x \in \mathbb{R}$$

(A4) Inverse under addition

$$\text{For any } x \in \mathbb{R}, \text{there exists } -x \in \mathbb{R} \text{ such that } x + (-x) = 0 = (-x) + x$$

(A5) Commutative under addition

$$\text{For any } x \in \mathbb{R} \text{ and } y \in \mathbb{R}, \quad x + y = y + x$$

Axioms for Multiplication

(M) $(\mathbb{R}, \cdot)$ is not (but almost be) a commutative group under multiplication

(M1) Closed under multiplication

$$\text{For any } x \in \mathbb{R} \text{ and } y \in \mathbb{R}, \quad xy \in \mathbb{R}$$

(M2) Associative under multiplication

$$\text{For any } x \in \mathbb{R}, y \in \mathbb{R} \text{ and } z \in \mathbb{R}, \quad (xy)z = x(yz)$$

(M3) Identity of multiplication

$$\text{There exists } 1 \in \mathbb{R} \text{ such that } 1x = x = x1 \text{ for all } x \in \mathbb{R}$$

(M4) Inverse under multiplication except 0

$$\text{For any } x \in \mathbb{R}\backslash\{0\}, \text{there exists } \frac{1}{x} \in \mathbb{R} \text{ such that } x\left(\frac{1}{x}\right) = 1 = \left(\frac{1}{x}\right)x$$

(M5) Commutative under multiplication

$$\text{For any } x \in \mathbb{R} \text{ and } y \in \mathbb{R}, \quad xy = yx$$

- $(\mathbb{R}, +, \cdot)$ is a *Field*
  - (A) Axioms for Addition: $(\mathbb{R}, +)$ satisfies (A1) – (A5)
  - (M) Axioms for Multiplicaiton: $(\mathbb{R}, \cdot)$ satisfies (M1) – (M5)
  - (D) The Distributive Law: For any $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $z \in \mathbb{R}$

$$x(y+z) = xy + xz \quad \text{and} \quad (x+y)z = xz + yz$$

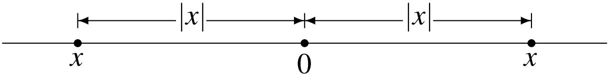
- Let $\leq$ be an order in $\mathbb{R}$ and $<$ be a strict order in $\mathbb{R}$
  1. x is called *positive* if $x > 0$ and x is called *negative* if $x < 0$
  2. x is called *nonnegative* if $x \geq 0$ and x is called *nonpositive* if $x \leq 0$
- $(\mathbb{R}, +, \cdot, <)$ is an *Ordered Field*
  1. If $x, y, z \in \mathbb{R}$ and $y < z$, then $x + y < x + z$
  2. If $x > 0$ and $y > 0$, then $xy > 0$

Definition

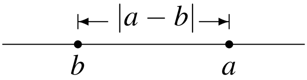
For any $x \in \mathbb{R}$, the *absolute value* of x, denoted by $|x|$, is defined by

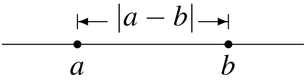
$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

- Always $|x| \geq 0$. In particular $|x| = 0$ if and only if $x = 0$.
- Intuitively $|x|$ means the "distance" from 0 to x in real line.



- Moreover, $|a - b| = |b - a|$ intuitively means the "distance" between a and b.





Convex Combination

Definition

Let a and b be any two points and $0 \leq w \leq 1$. The point $wa + (1 - w)b$ is called the *convex combination* between a and b.

- Intuitively, the convex combination between a and b is a point which is a "weighted average" between a and b.
- The weight is $w \in [0, 1]$. The higher the weight w, the closer the convex combination to point a.

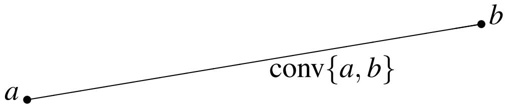
$w = 1$	$w = \frac{3}{4}$	$w = \frac{1}{2}$	$w = \frac{1}{4}$	$w = 0$
a	$\frac{3}{4}a + \frac{1}{4}b$	$\frac{1}{2}a + \frac{1}{2}b$	$\frac{1}{4}a + \frac{3}{4}b$	b

Definition

Let a and b be any two points. The *convex hull* between a and b, denoted by $\text{conv}\{a, b\}$, is the set of all convex combinations between a and b

$$\text{conv}\{a, b\} = \{wa + (1 - w)b \mid 0 \leq w \leq 1\}$$

Intuitively, the convex hull between a and b is the line segment connecting a and b.



Convex Set

Definition

Let D be a set. D is called *convex set* if for any $a \in D$ and $b \in D$, the convex combination $wa + (1 - w)b \in D$ for all $0 \leq w \leq 1$. That is D is convex set if $\text{conv}\{a, b\} \subset D$ for all $a, b \in D$.

- Intuitively, a convex set is a set which every possible line segment connecting any two points in the set must lie inside the set.
- Examples of convex set

- Examples of nonconvex set

Integer Power

Definition

For any $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

1. Define $a^n = \underbrace{a \cdots a}_{n \text{ terms}}$.
2. If $a \neq 0$, define $a^{-n} = \frac{1}{a^n}$.
3. If $a \neq 0$, define $a^0 = 1$.

Note that 0^0 is undefined in mathematics.

Theorem

Let $m, n \in \mathbb{Z}$ and $a, b \in \mathbb{R}$ such that their powers are well-defined.

1. $a^{m+n} = a^m a^n$
2. $a^{mn} = (a^m)^n = (a^n)^m$
3. $(ab)^n = a^n b^n$

Definition

For any $a > 0$ and $n \in \mathbb{N}$, the n^{th} *root* of a is a positive real number $a^{\frac{1}{n}}$ (or denoted by $\sqrt[n]{a}$) such that $\left(a^{\frac{1}{n}}\right)^n = a$.

Since a rational number $q \in \mathbb{Q}$ is a ratio of some two integers, so it can be written as $q = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ and $n > 0$. Hence the rational power can be defined.

Definition

For any $a > 0$ and $m, n \in \mathbb{Z}$ and $n > 0$, define $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$.

Since a real number is a limit of rational numbers.

Definition

For any $a > 0$ and $r \in \mathbb{R}$, define $a^r = \lim\limits_{q \to r} a^q$ where q is rational numbers.

Theorem

Let $r, s \in \mathbb{R}$ and $a, b \in \mathbb{R}$ such that their powers are well-defined.

1. $a^{r+s} = a^r a^s$
2. $a^{rs} = (a^r)^s = (a^s)^r$
3. $(ab)^r = a^r b^r$

Factorial and Combination

Definition

For any $n \in \mathbb{N}$, define the factorial of n as $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. Furthermore, define $0! = 1$.

The number $n!$ represents the number of all possible permutations of n distinct objects.

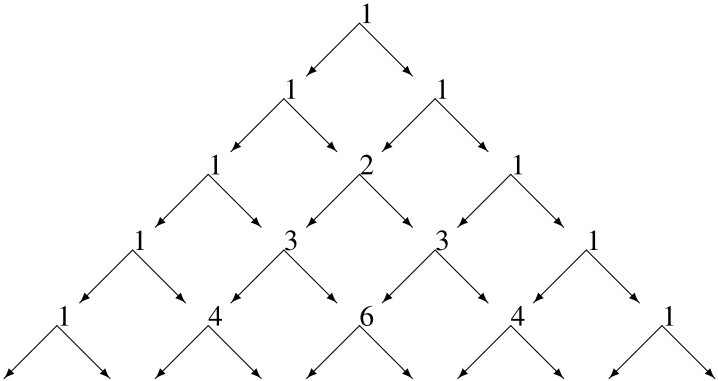
Definition

For any two nonnegative integers $n \geq r \geq 0$, define the combination as

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

- The number $\binom{n}{r}$ represents the number of all possible combinations of choosing r objects from n distinct objects.
- Note also that $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{1} = \binom{n}{n-1} = n$.

Pascal's Triangle



Each interger in the Pascal's triangle is the sum of the two integers above.

Pascal's Triangle and Combinatorics

Surprisingly, the number in the Pascal's triangle corresponds to the combinations.

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Theorem (Pascal's Triangle)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The Binomial Theorem

Consider the following expansions:

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= 1a + 1b \\
(a+b)^2 &= 1a^2 + 2ab + 1b^2 \\
(a+b)^3 &= 1a^3 + 3a^2b + 3ab^2 + 1b^3 \\
(a+b)^4 &= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4
\end{aligned}$$

The cofficients of the expansion corresponds to the number in the Pascal's triangle.

Theorem (Binomial Theorem)

Let $a, b \in \mathbb{R}$ *and* $n \in \mathbb{N} \cup \{0\}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

- Informally, we may think a sequence as a set of numbers put in order. We denote a_n as the n^{th} number in the sequence. So the sequence can be represented by a set $\{a_n\}_{n=1}^{\infty}$.
- Example of sequences:
  1. $1, 2, 3, 4, 5, \ldots$
  2. $2, 4, 6, 8, 10, \ldots$
  3. $1, 2, 4, 8, 16, 32, \ldots$
  4. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
- Many times we can identify the sequence by a simple relationship between its order (n) and its value (a_n). For example:
  1. $1, 2, 3, 4, 5, \ldots$ can be represented by $a_n = n$
  2. $2, 4, 6, 8, 10, \ldots$ can be represented by $a_n = 2n$
  3. $1, 2, 4, 8, 16, 32, \ldots$ can be represented by $a_n = 2^{n-1}$
  4. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ can be represented by $a_n = 2^{-n}$

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ is called an *arithmetic sequence* if

$$a_{n+1} - a_n = d$$

for some $d \in \mathbb{R}$ and for all $n \in \mathbb{N}$.

- Arithmetic sequence is a sequence that any two adjacent terms in the sequence has the same difference d.
- The number d is called the "common difference" of the arithmetic sequence.

Arithmetic Sequence: Formula

For arithmetic sequence, let a_1 be the first number in the sequence. If we know the common difference d, we can iterate to find other terms in the sequence by

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_2 + d = (a_1 + d) + d = a_1 + 2d \\ a_4 &= a_3 + d = (a_1 + 2d) + d = a_1 + 3d \quad \text{and so on} \end{aligned}$$

Theorem

An arithmetic sequence with common difference d can be represented by

$$a_n = a_1 + (n-1)d$$

Geometric Sequence: Definition

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ is called a *geometric sequence* if

$$\frac{a_{n+1}}{a_n} = r$$

for some $r \in \mathbb{R}$ and for all $n \in \mathbb{N}$.

- Geometric sequence is a sequence that any two adjacent terms in the sequence has the same ratio r.
- The number r is called the "common ratio" of the geometric sequence.

Geometric Sequence: Formula

For geometric sequence, let a_1 be the first number in the sequence. If we know the common ratio r, we can iterate to find other terms in the sequence by

$$\begin{aligned} a_2 &= a_1 r \\ a_3 &= a_2 r = (a_1 r) r = a_1 r^2 \\ a_4 &= a_3 r = (a_1 r^2) r = a_1 r^3 \quad \text{and so on} \end{aligned}$$

Theorem

A geometric sequence with common ratio r can be represented by

$$a_n = a_1 r^{n-1}$$

Definition

The partial sums of a sequence a_n is a sequence of the summation of the first N terms of that sequence.

$$S_N = \sum_{n=1}^{N} a_n$$

From the definition, the partial sums is a sequence $\{S_N\}_{N=1}^{\infty}$ where

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ S_4 &= a_1 + a_2 + a_3 + a_4 \quad \text{and so on} \end{aligned}$$

Partial Sums of Arithmetic Sequence

Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be an arithmetic sequence with common difference d. The partial sums is

$$S_N = Na_1 + \frac{N(N-1)}{2}d$$

Proof.

$$\begin{aligned} S_N &= \sum_{n=1}^{N} a_n = \sum_{n=1}^{N} \big(a_1 + (n-1)d\big) = a_1 N + d\sum_{n=1}^{N} n - dN \\ &= a_1 N + d\left(\frac{N(N+1)}{2}\right) - dN = a_1 N + \frac{N(N-1)}{2}d \end{aligned}$$

$\square$

Partial Sums of Geometric Sequence

Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be a geometric sequence with common ratio r. The partial sums is

$$S_N = \frac{a_1(1-r^N)}{1-r}$$

Proof.

$$\begin{aligned} S_N &= a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{N-1} \quad (1) \\ rS_N &= a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^N \quad (2) \end{aligned}$$

(1) − (2) gives $S_N - rS_N = a_1 - a_1 r^N$. So $S_N = \dfrac{a_1(1-r^N)}{1-r}$ □

Definition

A series s is a limit of the partial sums. $s = \lim_{N\to\infty} S_N$.

Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be a geometric sequence with common ratio $|r| < 1$. The geometric series is $s = \dfrac{a_1}{1-r}$.

Proof.

If $|r| < 1$ then $\lim_{N\to\infty}(1 - r^N) = 1 - 0 = 1$, hence

$$s = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a_1(1-r^N)}{1-r} = \frac{a_1}{1-r}$$

$\square$

Definition

The number e is defined by

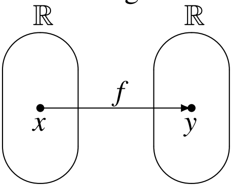
$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

- The number e is defined as a series of a sequence $\{a_n\}_{n=0}^{\infty} = \left\{ \frac{1}{n!} \right\}_{n=0}^{\infty}$.
- The number e is an irrational number, approximately $e \approx 2.71828$.

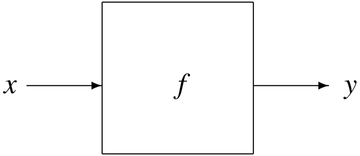
Theorem

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n$$

- A function f is a "mapping" from an element in a set to an element in another (could be the same) set.
- The set which is "mapped from" is called the "domain" of function, denoted by $\mathscr{D}_f$.
- The set which is "mapped to" is called the "codomain" of function.
- If function f maps from $\mathbb{R}$ to $\mathbb{R}$, we write $f : \mathbb{R} \to \mathbb{R}$.
- If an element x is mapped to an element y, we write $x \overset{f}{\mapsto} y$. In general, we may write an element y as $f(x)$; hence $x \mapsto f(x)$.
- The set of all y which is mapped from any element $x \in \mathscr{D}_f$ is called the "range" of function, denoted by $\mathscr{R}_f$.



$f : \mathbb{R} \to \mathbb{R}$



$x \overset{f}{\mapsto} y$

Injective/Surjective/Bijective Functions

Definition

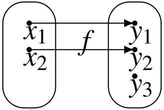
A function f is called "injective" or "one-to-one" if for any $y \in \mathscr{R}_f$, there is only one $x \in \mathscr{D}_f$ such that $y = f(x)$, i.e., every x and y is paired one-to-one.

Definition

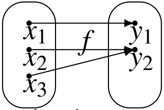
A function f is called "surjective" or "onto" if its range $\mathscr{R}_f$ equals to its codomain, i.e., all the elements in the codomain is mapped.

Definition

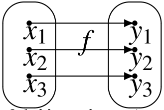
A function f is called "bijective" or "one-to-one onto" if they are both injective (one-to-one) and surjective (onto).



f injective (one-to-one)



f surjective (onto)

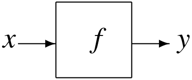


f bijective (one-to-one onto)

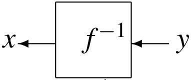
Inverse Function

Definition

Let $f : A \to B$ be bijective function, the inverse function $f^{-1} : B \to A$ is defined by $f^{-1}(y) = \{x \in A \mid y = f(x)\}$ for all $y \in B$.



$x \overset{f}{\mapsto} y$

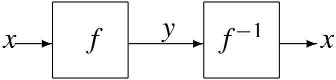


$y \overset{f^{-1}}{\mapsto} x$

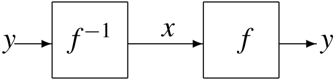
Theorem

Let f be a bijective function whose inverse is f^{-1}.

1. $(f^{-1} \circ f)(x) = f^{-1}\big(f(x)\big) = x$
2. $(f \circ f^{-1})(y) = f\big(f^{-1}(y)\big) = y$



$(f^{-1} \circ f)(x) = x$



$(f \circ f^{-1})(y) = y$

Monomial Function

- Constant function: A function f which maps every element x in the domain to the same element c in the codomain. Hence, the constant function is $f(x) = c$ for all x.
  - Examples: $f(x) = 0, f(x) = -1, f(x) = e$, etc.
  - Constant function $f(x) = c$ gives a horizontal line graph with y-intercept at c.
- Monomial function: A function f of the form $f(x) = ax^n$ where $n \in \mathbb{N} \cup \{0\}$ and $a \neq 0$ when $n \neq 0$.
  - a is called the coefficient of the monomial.
  - n is called the degree of monomial.
  - The monomial degree 0 function $(n = 0)$ is a constant function $f(x) = a$.
  - Example: $f(x) = 3x^2$ is a monomial degree 2 function with coefficient 3.

Polynomial Function

Polynomial function: A summation of monomial functions

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $n \in \mathbb{N} \cup \{0\}$ and $a_n \neq 0$ when $n \neq 0$.

- $a_0, a_1, \ldots, a_N$ are the coefficients of the polynomial.
- n is the degree of polynomial.
- Example: $f(x) = 3x^2 + 2x + 1$ is a polynomial degree 2 function.
- The polynomial degree 0 function is a constant function $f(x) = a_0$.
- The polynomial degree 1 function is an affine function $f(x) = mx + b$ which gives a linear graph with slope m and y-intercept b.
- The polynomial degree 2 function is called a quadratic function.
- The polynomial degree 3 function is called a cubic function.

1. Know the slope of the graph m and a point (x_0, y_0) on the graph.
    - $y = mx + b$, so $y_0 = mx_0 + b$. Hence $b = y_0 - mx_0$. Therefore,

$$y = mx + (y_0 - mx_0)$$

2. Know two points (x_1, y_1) and (x_2, y_2) on the graph.
    - First calculate the slope $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$.
    - Now use the above method to obtain

$$\begin{aligned} y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + \left(y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1\right) \\ \text{or} \quad y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + \left(y_2 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_2\right) \end{aligned}$$

Quadratic Function $f(x) = ax^2 + bx + c$.

- The function is *convex function* if $a > 0$ and it is *concave function* if $a < 0$.
- The peak (or trough) is at $x^* = -\dfrac{b}{2a}$.
- The x-intercept is at x_0 where $ax_0^2 + bx_0 + c = 0$

$$\begin{aligned}
x_0^2 + \frac{b}{a}x_0 + \frac{c}{a} &= 0 \\
x_0^2 + \frac{b}{a}x_0 + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} &= 0 \\
\left(x_0 + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} &= 0 \\
\left(x_0 + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 &= 0
\end{aligned}$$

Quadratic Formula

$$\left(x_0 + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) \qquad \left(x_0 + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0$$

$$x_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac > 0$ then there are two real solutions. The graph has two x-intercepts.
- If $b^2 - 4ac = 0$ then $x_0 = \dfrac{-b}{2a} = x^*$. Hence, the peak (or trough) is the only point touching the x-axis.
- If $b^2 - 4ac < 0$ then x_0 are not real numbers (they are two complex conjugate numbers). Hence, the graph is totally above the x-axis (if $a > 0$) or it is totally below the x-axis (if $a < 0$).

Rational Function and Algebraic Function

- Rational function: A ratio of two polynomials

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x_1 + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}$$

  where p is polynomial degree m function and q is polynomial degree n function.

  - The degree of rational function is the degree of the numerator polynomial minus the degree of the denominator polynomial.
  - Example: $f(x) = \dfrac{2x+1}{3x^2+x+2}$ is a rational function degree $1-2=-1$.

- Algebraic function: An algebraic completion of rational functions.

  - Examples: $f(x) = \sqrt{x}, f(x) = \dfrac{x^{\frac{1}{3}}+1}{x^{\frac{2}{3}}+2x^{\frac{1}{3}}+2}$, etc.

Transcendental function is a function which is not algebraic functions. Examples are

- Exponential functions: $f(x) = e^x, f(x) = a^x, f(x) = e^{2x}$, $f(x) = e^{-x^2/2}$, etc.
- Logarithm functions: $f(x) = \log(x), f(x) = \log_a(x)$, etc.
- Trigonometric functions: $f(x) = \sin(x), f(x) = \cos(x)$, $f(x) = \tan(x)$, etc.
- Inverse trigonometric functions: $f(x) = \arcsin(x)$, $f(x) = \arccos(x), f(x) = \arctan(x)$, etc.
- Hyperbolic trigonometric functions: $f(x) = \sinh(x)$, $f(x) = \cosh(x), f(x) = \tanh(x)$, etc.

Definition

The exponential function $f(x) = e^x$ (or $\exp(x)$) is a function defined by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Another useful form of the exponential function is

Theorem

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

Definition

For $a > 0$ and $a \neq 1$, the exponential base a function is defined by $f(x) = a^x$

- When base $a = e$, it is the ordinary exponential function.
- $f(x) = a^x$ is always strictly positive, i.e., the range of function $\mathscr{R}_f = \mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$.
- If $a > 1$ the function is strictly increasing.
- If $0 < a < 1$ the function is strictly decreasing.
- Since the exponential base a function is stricly increasing/decreasing (hence one-to-one mapping) onto its range, so the inverse of exponential base a function is a function.

Definition

The logarithm function $f : \mathbb{R}_{++} \to \mathbb{R}$, denoted by $f(x) = \log(x)$ $\big($or $\ln(x)\big)$, is defined as the inverse function of the exponential function. In particular,

$$y = \log(x) \quad \text{if and only if} \quad x = e^y$$

Theorem

1. $\log(ab) = \log(a) + \log(b)$
2. $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$
3. $\log(a^x) = x\log(a)$ *(*$\log$ *is an inverse of* $\exp$*)*
4. $e^{a\log(x)} = x^a$ *(*$\exp$ *is an inverse of* $\log$*)*

Logarithm Base a Function

Definition

When $a > 0$ and $a \neq 1$, the logarithm base a function, denoted by $f(x) = \log_a(x)$, is defined as the inverse function of the exponential base a function.

$$y = \log_a(x) \quad \text{if and only if} \quad x = a^y$$

The relationship between the ordinary logarithm and logarithm base a is

Theorem

$$\log_a(x) = \frac{\log(x)}{\log(a)} \quad \textit{or equivalently} \quad \log(x) = \frac{\log_a(x)}{\log_a(e)}$$

- Initial deposite S_0. Interest rate is r per annum.
- Annually compound:

$t = 0$ $\qquad$ $t = 1$

S_0 $\qquad$ $S_1 = S_0(1 + r)$

- Half-annually compound:

$t = 0$ $\qquad$ $t = \frac{1}{2}$ $\qquad$ $t = 1$

S_0 $\qquad$ $S_{\frac{1}{2}} = S_0\left(1 + \frac{r}{2}\right)$ $\qquad$ $S_1 = S_0\left(1 + \frac{r}{2}\right)^2$

- Quarterly compound:

$t = 0$ $\qquad$ $t = \frac{1}{4}$ $\qquad$ $t = \frac{2}{4}$ $\qquad$ $t = \frac{3}{4}$ $\qquad$ $t = 1$

S_0 $\qquad$ $S_0\left(1 + \frac{r}{4}\right)$ $\qquad$ $S_0\left(1 + \frac{r}{4}\right)^2$ $\qquad$ $S_0\left(1 + \frac{r}{4}\right)^3$ $\qquad$ $S_1 = S_0\left(1 + \frac{r}{4}\right)^4$

Continuous Compounding

- If deposite for T years

Compound	$t=0$	$t=1$	$t=T$
Annually	S_0	$S_0(1+r)$	$S_0(1+r)^T$
Half-Annually	S_0	$S_0\left(1+\frac{r}{2}\right)^2$	$S_0\left(1+\frac{r}{2}\right)^{2T}$
Quarterly	S_0	$S_0\left(1+\frac{r}{4}\right)^4$	$S_0\left(1+\frac{r}{4}\right)^{4T}$
Monthly	S_0	$S_0\left(1+\frac{r}{12}\right)^{12}$	$S_0\left(1+\frac{r}{12}\right)^{12T}$
Daily	S_0	$S_0\left(1+\frac{r}{365}\right)^{365}$	$S_0\left(1+\frac{r}{365}\right)^{365T}$
n times per year	S_0	$S_0\left(1+\frac{r}{n}\right)^n$	$S_0\left(1+\frac{r}{n}\right)^{nT}$

- If we let $n \to \infty$, we will have *continuously compound*

$$S_1 = \lim_{n\to\infty} S_0\left(1+\frac{r}{n}\right)^n = S_0 \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = S_0 e^r$$

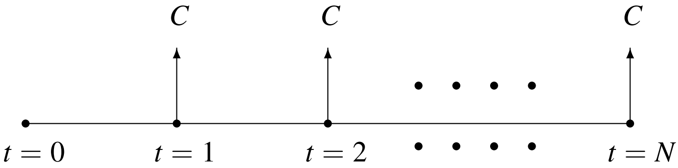
- Therefore, after T years, the balance will be

$$S_T = S_0\left(e^r\right)^T = S_0 e^{rT}$$

- As long as the risk-free rate of return is not zero, the money today can be invested to obtain interest in the future.
- Hence S_0 Baht today can be $S_0(1 + r)$ next year. Therefore, investor prefer having S_0 Baht today to having S_0 Baht next year.
- This means the money at different time has different value (as long as the interest is not zero). This pheonomenon is called "the time value of money".
- Time value of money depends only on the interest rate, it is irrelevant to the price of consumption goods. Hence, time value of money and inflation are different topics.
- Given that investors will receive a stream of certain payoffs in the future, this stream of incomes in the future can be evaluate its value today. Its equivalent value today is called "net present value (NPV)".

Net Present Value (Finite Stream of Payoffs)

- Suppose an investor has an asset which will pay C Baht every year for N years.

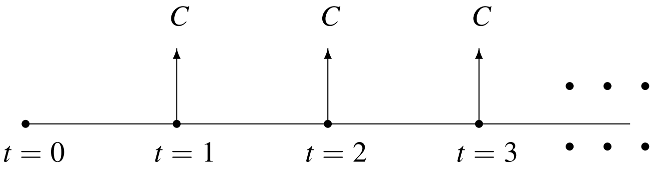


- The NPV can be calculated by partial sum of geometric sequence with common ratio $= \dfrac{1}{1+r}$

$$
\begin{aligned}
NPV &= \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^N} \\
&= \frac{\frac{C}{1+r}\left(1 - \frac{1}{(1+r)^N}\right)}{1 - \frac{1}{1+r}} = \frac{C\big((1+r)^N - 1\big)}{r(1+r)^N}
\end{aligned}
$$

Net Present Value (Infinite Stream of Payoffs)

- Suppose an investor has an asset which will pay C Baht every year indefinitely.

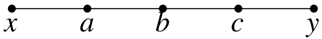


- The NPV can be calculated by geometric series with common ratio $= \dfrac{1}{1+r}$

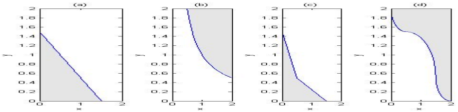
$$
\begin{aligned}
NPV &= \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \\
&= \frac{\frac{C}{1+r}}{1 - \frac{1}{1+r}} = \frac{C}{r}
\end{aligned}
$$

Exercise I

1. Let x and y be two real numbers. The interval $[x, y]$ is devided into four parts equally as in the figure below. Write a, b and c in the form of convex combination between x and y.



2. Consider the sets represented by the shaded area of in the figures below. Which one is convex set and which one is not?



3. Expand $(a + b)^7$.
4. Let $f(x) = 3 - 27x^3$. Compute $f(0), f(-1), f(1/3)$ and $f(\sqrt[3]{2})$.

Exercise II

1. Determine the equations for the following straight lines:
    1. L_1 passes through $(-2, 3)$ and has a slope of -3.
    2. L_2 passes through $(-3, 5)$ and $(2, 7)$.
2. Simplify the following expressions:
    1. $e^{\ln(x)} - \ln\left(e^x\right)$.
    2. $\ln\left(x^4 e^{-x}\right)$.
    3. $e^{\ln\left(x^2\right) - 2\ln y}$.
3. How much does \$8000 grow to after 5 years if the annual interest rate is 5%, with continuous compounding? How long does it take before the \$8000 has double?
4. Suppose that the interest rate is 5% per year. What is the NPV of the coupon bond which pays 500 Baht at the end of each year for 20 years? What is the NPV of this coupon bond if it pays forever? $(1.05^{20} \approx 2.6533.)$

Exercise III

Susan is a project engineer, she is assigned to analyze two alternatives of an investment plan. They are:

Plan A:	Buy a new-release personal computer that costs \$10,000. The new-release personal computer's life is 4 years. The return will be received \$5,000 at the end of each year. The interest rate is given at 2% a year.
Plan B:	Buy a refurbished personal computer costs \$7,000 now. The refurbished personal computer's life is 2 years. At the end of year 2, buy a new PC that will cost \$11,000. The second PC's life is also 2 years. The returns for this project at the end of each year are \$4,000, \$4,000, \$6,000 and \$6,000 respectively. The interest rate is given at 2% a year.

Exercise III (Continue)

1. What is the present value of the return in plan A?
2. What is the NPV of plan A?
3. What is the present value of the returns in plan B?
4. What is the total cost (the current cost + the present value of the cost that will incur in the future) in plan B?
5. What is the NPV of plan B?
6. Which project plan will Susan undertake?